effect when the bubbles are small is borne out by observation, but it is difficult to make quantitative experiments in these cases.

When the proportion of air to water is large the mixture becomes a froth that is a mass of polygonal bubbles separated by thin films of liquid. The problem of evaluating the viscous reaction in such a mixture is one I have not attempted to solve. Experiment, however, shows that froth is a most efficient agent in damping vibrations.

This may be shown in a simple and striking way as follows:—Having prepared a sufficient quantity of fine grained froth in a flat dish with soap and water, take a wine glass or tumbler which gives a clear musical note when struck. Dip the mouth of the glass into the froth for about a quarter of an inch and withdraw it. A thin ring of froth will be left round the edge, weighing, perhaps, 1 or 2 grains. Small as this quantity of froth is it will be found sufficient to damp the vibration so rapidly as to almost deprive the sound of any musical character.

On the Determination of the Tension of a recently formed Water-Surface.

By N. Bohr, Copenhagen.

(Communicated by Lord Rayleigh, O.M., F.R.S. Received August 22,—
Read November 10, 1910.)

As an addition to my paper, published in the 'Philosophical Transactions,'* on the determination of the surface-tension of water, I desire to set forth the following remarks concerning the problem of the value of the tension of a recently formed water-surface, and the circumstances which are of importance for the determination of this tension.

Prof. P. Lenard has, in a paper lately published,† determined the surface-tension of a recently formed water-surface by means of the vibration of falling drops, and has for this tension found values which are considerably greater than those found by other methods. From this, as well as from the results of experiments published in a former paper,‡ he concludes that a recently formed water-surface has a very great tension, which, however, in the course of a very short time (fraction of a second) decreases considerably. He remarks that this result is in agreement with experiments published in my paper mentioned above. I shall, however, in the following try to explain the reasons why I cannot agree in these conclusions.

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* 'Phil. Trans.,' A, 1909, vol. 209, p. 281.

† 'Sitzungsber. d. Heidelberger Akad. d. Wiss.,' Math.-nat. Kl. Jahrg. 1910, Abh. 18.

‡ 'Ann. d. Phys. u. Chem.,' 1887, vol. 30, p. 209.

The determination of the surface-tension published in my paper cited above was carried out by the method of jet-vibrations, the theoretical foundation of which method, as well as of the method used by Prof. Lenard, is due to Lord Rayleigh.*

As to the closer investigation of the vibration of the jets, especially with regard to the satisfaction of the suppositions made in the theoretical treatment of the phenomenon, a great number of vibrations, commencing just at the orifice and in the most stable jets extending to a distance of more than 45 cm. from this (the velocity of the jets was about 425 cm./sec.), were examined by my experiments. These measurements showed that the wave-length was not the same everywhere, but that, advancing from the orifice, it increased in the beginning rather rapidly and thereupon more slowly until finally from a distance of about 25 cm. from the orifice and as far as the stability of the jet allowed the measurings, the wave-length became practically perfectly constant. (See the tables, *loc. cit.*, pp. 310—312.)

This result consequently showed the existence of certain irregularities of the phenomenon, which arise in the formation of the jet, and which are rapidly (in about 0·06 sec.) extinguished (*loc. cit.*, p. 309).

These irregularities might partly be thought to originate from possible variations of the value of the surface-tension in the time immediately after the formation of the surface, partly from irregularities of a more mechanical (hydrodynamical) character (*loc. cit.*, p. 299). Since the last-mentioned irregularities, as explained in my former paper, must decrease rapidly on removal from the issue of the jet, the result of the experiments showed that the surface-tension, in every case from about 0·06 second after formation of the surface, and as long as it was possible to investigate the tension by the used method, was sensibly constant.† This constant value was considered as the sought value for the surface-tension, and it was in every case the only one which could be compared with values found by other methods, in which the investigated surfaces always have been much older than 0·06 second.

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* 'Roy. Soc. Proc.,' 1879, vol. 39, p. 71.

† Prof. Lenard remarks in his last paper (*loc. cit.*, p. 4) that the mechanical irregularities certainly must decrease, but cannot, even far from the orifice, completely disappear, on account of the resistance of the air against the movement of the jet. As, however, the effect of the air resistance removing from the orifice very rapidly will become constant, we see that an influence of this resistance on the phenomenon will not affect the above conclusion of the constancy of the surface-tension, but it can only cause an alteration of the value found for this constant tension. As to the question of the magnitude of the influence of the air resistance, I mention here an unpublished experiment made during my first investigation. Around the jet, at a distance of about 10 cm. from the orifice, was placed a large and carefully worked iris diaphragm, so

Concerning the question of a possible variation of the surface-tension during the time from the formation of the surface until some 0·06 second later, it seems to me that my experiments do not give any reason to conclude an existence of such a variation, there being, as we shall see, no objection in explaining the found variation in the wave-length by help of the velocity-differences between concentric parts of the jet produced by friction during its formation, by which the central parts receive a greater velocity than the parts nearer the surface. These velocity-differences decrease, removing from the issue, on account of the viscosity in the jet; the mean velocity of the jet keeping constant, this effects that the velocity of the outer parts increases at the same time as the velocity of the central parts decreases. That the wave-lengths are shorter close to the orifice than at a greater distance from this has always seemed to me to be a natural consequence of the velocity of the surface (the outer parts) here being smaller and the waves in question being surface-waves (the velocity of the vibrating liquid-particles is becoming smaller removing from the surface, and is vanishing in the axis of the jet). In his above-cited paper (*loc. cit.*, p. 4), Prof. Lenard, however, is of the opinion that the inner mixture—produced during the vibrating motion, on account of the mutual displacing of the concentric parts of the jet—will effect an apparent increase of the mass, and a thereby resulting prolongation of the time of vibration and increase of the wave-lengths.

For the closer examination of this question, I have therefore made the following direct calculation of the wave-lengths, under the assumption that the different concentric parts of the jet are moving with different velocities.

that the jet just passed through the centre of the diaphragm. This was at first open, so that a free space of 5 cm. (the opening of the diaphragm was 10 cm.) surrounded the jet. Thereupon the diaphragm was closed, so that the free space between the jet and the diaphragm was not more than some 0·2 mm., and at the same time a wave-summit of the jet at a distance of 30 cm. from the orifice was fixed in a telescope by help of reflection in the surface of the jet (*loc. cit.*, p. 305). It was then observed that the mentioned summit during the closing of the diaphragm was displaced only very little (less than 0·1 mm.). This simple experiment was repeated several times with exactly the same result. As such a closing of the diaphragm must increase the resistance of the air to a very considerable degree, completely stopping the mass of air set in motion by the jet (the jet produces a sensible blast), the experiment, in my opinion, shows very distinctly that the air resistance cannot have any appreciable influence on the results. As will be shown in the following, an air resistance would besides introduce a correction of the value of the surface-tension, the sign of which would be opposite to that supposed by Prof. Lenard.

The general equations of motion of an incompressible non-viscous fluid unaffected by extraneous forces, are

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \qquad \rho \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \qquad \rho \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}, \tag{1}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2}$$

in which u, v, w are the components of the velocity, p the pressure, ρ the density, and

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

In the problem in question the motion will be steady. Putting $w = \mathrm{W} + \omega$, and supposing that u, v, and ω are so small that products of them, and quantities of the same order of magnitude, can be neglected in the calculations, we get from the equations (1)

$$\rho \mathrm{W} \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}, \qquad \rho \mathrm{W} \frac{\partial v}{\partial z} = -\frac{dp}{dy}, \qquad \rho \left(u \frac{\partial \mathrm{W}}{\partial x} + v \frac{\partial \mathrm{W}}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z}. \tag{3}$$

Introducing polar co-ordinates r and ϑ ($x = r\cos\vartheta$, $y = r\sin\vartheta$), and the radial and tangential velocity α and β, we get by help of the relations

$$u = \alpha\cos\vartheta - \beta\sin\vartheta, \qquad v = \alpha\sin\vartheta + \beta\cos\vartheta,$$

from (3), assuming W to be a function of r only.

$$\rho \mathrm{W} \frac{\partial \alpha}{\partial z} = -\frac{\partial p}{\partial r}, \qquad \rho \mathrm{W} \frac{\partial \beta}{\partial z} = -\frac{1}{r}\frac{\partial p}{\partial \vartheta}, \qquad \rho \left(\alpha \frac{d\mathrm{W}}{dr} + \mathrm{W} \frac{\partial \omega}{\partial z} \right) = -\frac{\partial p}{\partial z}, \tag{4}$$

and from (2)

$$\frac{\partial \alpha}{\partial r} + \frac{\alpha}{r} + \frac{1}{r}\frac{\partial \beta}{\partial \vartheta} + \frac{\partial \omega}{\partial z} = 0. \tag{5}$$

Now, supposing that p, α, β, and ω have the form $f(r)\, e^{in\vartheta + ikz}$, we get from (4) and (5)

$$\frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{\partial r}\left(\frac{1}{r} - \frac{2}{\mathrm{W}}\frac{d\mathrm{W}}{dr} \right) - p\left(\frac{n^2}{r^2} + k^2 \right) = 0. \tag{6}$$

In the case W is constant, the solution of (6), subject to the condition to be imposed when $r = 0$, is

$$p_0 = \mathrm{AJ}_n(ikr)\, e^{in\vartheta + ikz}, \tag{7}$$

in which J_n is the symbol of the Bessel's function of nth order.

Putting

$$p = p_0 e^{\int_0^r \psi(r)\, dr}, \tag{8}$$

we get from (6)

$$\frac{d\psi}{dr} + \psi^2 + \psi\left(\frac{1}{r} + \frac{2}{p_0}\frac{\partial p_0}{\partial r} - \frac{2}{\mathrm{W}}\frac{d\mathrm{W}}{dr} \right) - \frac{2}{p_0 \mathrm{W}}\frac{\partial p_0}{\partial r}\frac{d\mathrm{W}}{dr} = 0. \tag{9}$$

We will now suppose that $W = c+\sigma$, in which the constant c is the mean velocity of the jet, and σ a quantity small compared with c. In this case ψ is small, and neglecting terms of the same order of magnitude as $(\sigma/c)^2$, we get from (9)

$$\frac{d\psi}{dr}+\psi\left(\frac{1}{r}+\frac{2}{p_0}\frac{\partial p_0}{\partial r}\right)-\frac{2}{cp_0}\frac{\partial p_0}{\partial r}\frac{d\sigma}{dr} = 0. \tag{10}$$

In the experiments the numerical value of ikr will be a very small quantity—the wave-length large in comparison to the diameter of the jet—in order not to complicate the formulæ, we will therefore in the calculation of ψ only use the first term of the expression for $J_n(ikr)$. This gives $\frac{1}{p_0}\frac{\partial p_0}{\partial r} = \frac{n}{r}$, and the solution of (8) becomes

$$\psi = \frac{2n}{c}r^{-(2n+1)}\left\{\int_0^r \frac{d\sigma}{dr}r^{2n}dr+\mathfrak{C}\right\}.$$

The motion being finite for $r = 0$, we have $\mathfrak{C} = 0$.

Integrating by part we get

$$\psi = \frac{2n}{cr}\sigma-\frac{4n^2}{cr^{2n+1}}\int_0^r \sigma r^{2n-1}dr. \tag{11}$$

Let us suppose that the equation of the surface is

$$r-a = \zeta = Be^{in\vartheta+ikz}.$$

The general surface condition gives

$$\frac{D}{Dt}(r-a-\zeta) = \left(\alpha\frac{\partial}{\partial r}+\frac{\beta}{r}\frac{\partial}{\partial\vartheta}+w\frac{\partial}{\partial z}\right)(r-a-\zeta) = 0,$$

whence we get, neglecting quantities of the same order of magnitude, as by the equations (3)

$$\alpha-W\frac{\partial\zeta}{\partial z} = 0, \qquad \zeta = -\frac{i}{Wk}\alpha.$$

In the same manner we get further, if the principal radii of curvature are R_1 and R_2,

$$\frac{1}{R_1}+\frac{1}{R_2} = \frac{1}{a}-\frac{\zeta}{a^2}-\frac{1}{a^2}\frac{\partial^2\zeta}{\partial\vartheta^2}-\frac{\partial^2\zeta}{\partial z^2} = \frac{1}{a}-\alpha\frac{i(n^2-1+k^2a^2)}{a^2Wk}.$$

Calling the surface-tension T, the dynamical surface-condition will be

$$T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-p = \text{const.}$$

From this we get, with the same approximation as before and using (4)

$$\left[T\frac{(n^2-1-k^2a^2)}{\rho a^2k^2W^2}\frac{\partial p}{\partial r}-p\right]_{r=a} = 0. \tag{12}$$

From (12) we get, using (7) and (8),

$$k^2 = T\frac{iakJ'_n(iak)}{\rho a^3J_n(iak)}(n^2-1+a^2k^2)\left\{\frac{1}{W^2}\left(1+\psi\frac{p_0}{\partial p_0/\partial r}\right)\right\}_{r=a} \tag{13}$$

From (13) we get, by help of (11) and with the same approximation as used in the calculation of ψ,

$$k = \mathrm{T}\frac{iak\mathrm{J}'_n(iak)}{\rho c^2 a^3 \mathrm{J}_n(iak)}(n^2-1+a^2k^2)\left[1-\frac{4n}{ca^{2n}}\int_0^a \sigma r^{2n-1}dr\right]. \tag{14}$$

This equation is, except the last term, the solution given by Lord Rayleigh. We therefore see that the effect of the velocity-differences between concentrical parts of the jet consists in an exchange in the formula for the wave-length λ ($\lambda = 2\pi/k$), of the mean velocity of the jet c by an "effective mean velocity,"

$$c' = c+\frac{2n}{a^{2n}}\int_0^a \sigma r^{2n-1}dr. \tag{15}$$

We see from (15) that the greater n is, the nearer the effective mean velocity will be to the velocity of the surface, which is explained by the fact that the greater n (the number of waves on the circumference of a section of the jet) is, the more rapidly the velocity of the vibrating liquid particles will decrease, moving from the surface towards the axis of the jet.

It can now be shown that c' will be smaller than c, if the velocity of the jet—which will be the case in the experiments—is greatest in the middle and continually decreases approaching the surface; c being the mean velocity of the jet, we have $\int_0^a \sigma r dr = 0$, and in the case in question further $\int_0^r \sigma r dr > 0$, where $a > r > 0$. From this we get for $n \geqq 2$ (in the experiments $n = 2$).

$$c'-c = \frac{2n}{a^{2n}}\int_0^a \sigma r^{2n-1}dr = -\frac{2n(2n-2)}{a^{2n}}\int_0^a\left[\int_0^r \sigma r dr\right]r^{2n-3}dr < 0. \tag{16}$$

After having seen that the velocity-differences in question will produce a variation of the wave-length in the same direction as found by the experiment, we shall further see how the decrement of the variation of the wave-length can also be approximately explained by the manner in which the velocity-differences will decrease. In order to show this, we shall use the four experiments quoted in the table, *loc. cit.*, p. 310. In the table below is quoted the mean radius of the jet a, the velocity v (calculated from the mean radius and the discharge), and under the indication $\mathrm{D}_2/\mathrm{D}_1$, the difference between the wave-length, measured between the wave-summits IV and V, and the constant value to which the wave-lengths were tending, divided by the corresponding difference between the wave-length, measured between the summits II and III, and the mentioned constant value (in these differences are introduced the small corrections for the curvature of the jet and for the wave-amplitudes mentioned in the table, *loc. cit.*, p. 311); under the indication l is quoted the difference between the mean value of the readings of the summits IV and V and the mean values of the readings II and III. Further, under the indication α is quoted the ratio between the variations of the wave-length in two places corresponding to a time-interval of 1/100 sec. (calculated on the assumption that the variations decrease after an exponential law). Under the indication α' is finally quoted the ratio between the velocity differences in the jet in two places corresponding to a time-interval of 1/100 sec., calculated from the theoretical formula, *loc. cit.*, p. 298, *l.* 2. f. b.

	I.	II.	III.	IV.
a	0·06755	0·07554	0·07595	0·08010
v	426	428	426	429
$\frac{D_2}{D_1}$	0·54	0·53	0·53	0·50
l	4·30	5·16	5·16	5·59
$\alpha = \left(\frac{D_2}{D_1}\right)^{\frac{v}{100\,l}}$	0·54	0·59	0·59	0·59
$\alpha' = e^{-\frac{\mu}{\rho}\left(\frac{\pi 1{\cdot}2197}{a}\right)^2 \frac{1}{100}}$	0·67	0·72	0·73	0·75

As will be seen, the calculated and the found values for the decrement of the variation of the wave-lengths agree with regard to the order of magnitude, and more was not to be expected from such an approximate calculation. It is thus not justified to expect that the distribution of the velocity in jet-sections so close to the orifice could be completely expressed by the first term in the general formula on p. 298.

After having now seen that my experiments do not give any reasons for the conclusion of the existence of a variation in the surface-tension during the first time after the formation of the surface, we shall proceed to mention the values for the surface-tension of a recently formed water-surface found by Prof. Lenard by his investigations of the vibrations of falling drops.

The surfaces there investigated must, in my opinion, be considered as much older than the surfaces investigated by my experiments, on account of the length of the time used for the formation of the drops. Prof. Lenard remarks in his last paper [*loc. cit.*, p. 11, note (18)], that this time—amounting from 0·17 to 1·05 sec. in his first paper, and from 0·6 to 0·9 sec. in his last—will contribute only a very little to the age of the surface of the drops, new surface continually being formed during this time. This circumstance does not, however, seem to me sufficiently to justify the neglecting of this, in this connection, very long time. I should rather be inclined to agree with the opinion set forth in his former paper, according to which the time for the formation of the drops is considered a measure for the age of the surface (*loc. cit.*, p. 233).

A comparison between the experiments of Prof. Lenard (*ibid.*, p. 236) and of Lord Rayleigh* on the surface-tension of a solution of soap seems also distinctly to show that the time of formation of the drops has a great influence on the condition of the surface. By the experiments of Lord Rayleigh with vibrating jets, the surface-tension of a solution of soap, 1/100 second after the formation of the surface, was thus found to be very near to that of pure water; while Prof. Lenard by experiments with vibrating drops (time of formation greater than $\frac{1}{4}$ sec.) finds the surface-tension of a soap solution of corresponding strength (1 : 1000) less than half that of water and rather near the stationary value of the surface-tension of a soap solution.

It appears, from the preceding, that the high values of the tension of a recently formed water-surface and the rapid decreasing of this value, which Prof. Lenard has found by his experiments, are not in agreement with the result of my previous experiments, because the tension of water-surfaces of lower age than those investigated by Prof. Lenard have been found much smaller and perfectly constant within the time interval (from 0·06 to 0·11 second after the formation of the surface) during which the method allowed the determination of the tension.

The cause of the great deviation between the results found by the method of drop-vibration and those found by the method of jet-vibration must, in my opinion, be sought in the circumstance that sufficient regard as to the influence of irregularities of mechanical character, arising from the disengagement of the drops, is scarcely taken by the drop method. The investigation of the influence of such irregularities seems also much more difficult by vibrating drops than by vibrating jets, the investigation by the latter being very much facilitated by the perfectly steady character of the phenomenon.

Concluding these remarks, I might call attention to the good agreement between the value of the tension of a water-surface 0·06 second old (73·23 dyne/cm. at 12° C.), found in my paper, and values of the tension of a water-surface found by statical methods† (Volkmann‡ 73·72 dyne/cm. at

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* 'Roy. Soc. Proc.,' 1890, vol. 47, p. 281.

† As to the result found by other methods, I might refer to the discussion in my former paper (*loc. cit.*, pp. 314—317).

‡ 'Wied. Ann.,' 1895, vol. 56, p. 457.

12° C., Eötvös* 73·06). This agreement seems to show that the tension of a water-surface already only 0·06 second after the formation of the surface (and according to what is discussed in the present paper probably much earlier) has assumed the constant value which the tension, if contaminations are kept away, will retain during a very long time.

The Tidal Observations of the British Antarctic Expedition, 1907.

By Sir George Darwin, K.C.B., F.R.S.

(Received September 27,—Read November 10, 1910.)

The present investigation was undertaken at the request of Sir Ernest Shackleton; the expense of the reduction was defrayed by him, and this paper is now communicated to the Royal Society by his permission. It will ultimately be republished as a contribution to the volume of the physical results of the expedition.

The first section, describing the method of observing, is by Mr. James Murray. The second section explains the reduction of the observations and gives a comparison between the new results and those obtained by the "Discovery" in 1902–3. The third section is devoted to the discussion of certain remarkable oscillations of mean sea-level and to speculations as to their cause and meaning.

I.—On the Method of Observing the Tides.

Early in June, 1908, preparations were begun for the erection of a tide-gauge, the most important feature of which was to be a recording apparatus made from a modified barograph. Owing to various delays and mishaps it was not before the middle of July that the gauge was completed in its final form, and the continuous record begun, which was carried on for more than three months, subject only to the loss of half an hour weekly, while the paper was being changed.

Dr. Mackay undertook the erection of the instrument, the apparatus was devised by the joint suggestions of Messrs. David, Mackay, Mawson, and Murray, while Mr. Day did the more delicate part of the work, namely, the alteration of the barograph.

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* 'Math. es Természettud.,' 1885, vol. 3, p. 54 (Budapest).